Question 1 [7 marks]

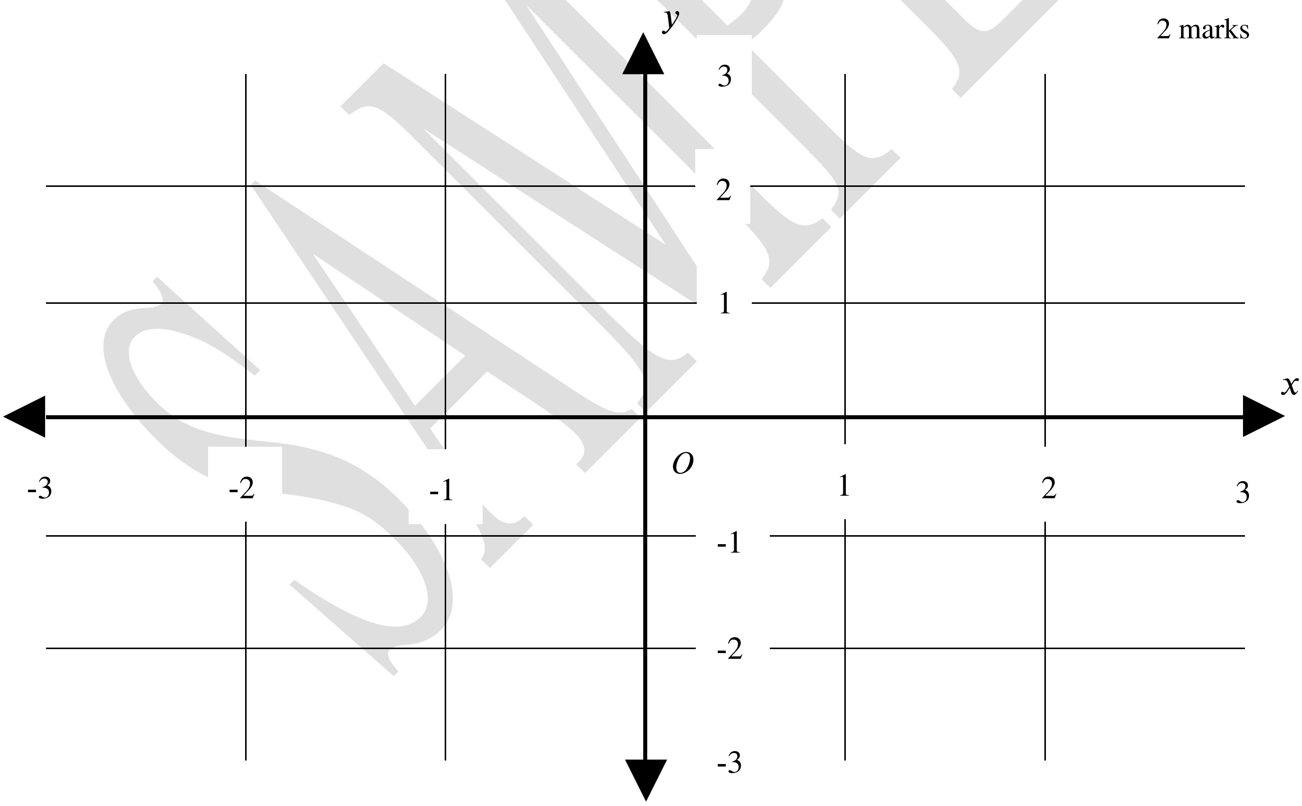
Let $f: R \to R, f(x) = x^3 - 2x$.

a. Find the coordinates of the turning points of $y = f(x)$.

2 marks

b. Sketch the graph of $y = f(x)$ on the axes below. Label any stationary points and axes intercepts with their coordinates.

2 marks



Let $g: R \to R, g(x) = x^3 - ax$ where $a \in R^+$.

c. Find the value(s) of a such that the line $y = x$ is tangent to the graph of g.

3 marks

Question 1 [7 marks]

Let $f: R \to R, f(x) = x^3 - 2x$.

d. Find the coordinates of the turning points of $y = f(x)$.

2 marks

Solution:

- Find the **derivative function.**

  $f'(x) = 3x^2 - 2$

- Solve $\boldsymbol{f'(x) = 0}$ for x.

  $3x^2 - 2 = 0$

  $x^2 = \frac{2}{3}$

  $x = \pm\frac{\sqrt{6}}{3}$

- Find the **values of $\boldsymbol{y = f(x)}$** when $x = \pm\frac{\sqrt{6}}{3}$.

  $$f\left(\frac{\sqrt{6}}{3}\right) = \left(\frac{\sqrt{6}}{3}\right)^3 - 2\left(\frac{\sqrt{6}}{3}\right)$$

  $$f\left(\frac{\sqrt{6}}{3}\right) = \frac{2\sqrt{6}}{9} - \frac{2\sqrt{6}}{3} = -\frac{4\sqrt{6}}{9}$$

  $$f\left(-\frac{\sqrt{6}}{3}\right) = \left(-\frac{\sqrt{6}}{3}\right)^3 - 2\left(-\frac{\sqrt{6}}{3}\right)$$

  $$f\left(-\frac{\sqrt{6}}{3}\right) = -\frac{2\sqrt{6}}{9} + \frac{2\sqrt{6}}{3} = \frac{4\sqrt{6}}{9}$$

- Write the turning points as **coordinates.**

  $\left(\frac{\sqrt{6}}{3}, -\frac{4\sqrt{6}}{9}\right)$ and $\left(-\frac{\sqrt{6}}{3}, \frac{4\sqrt{6}}{9}\right)$

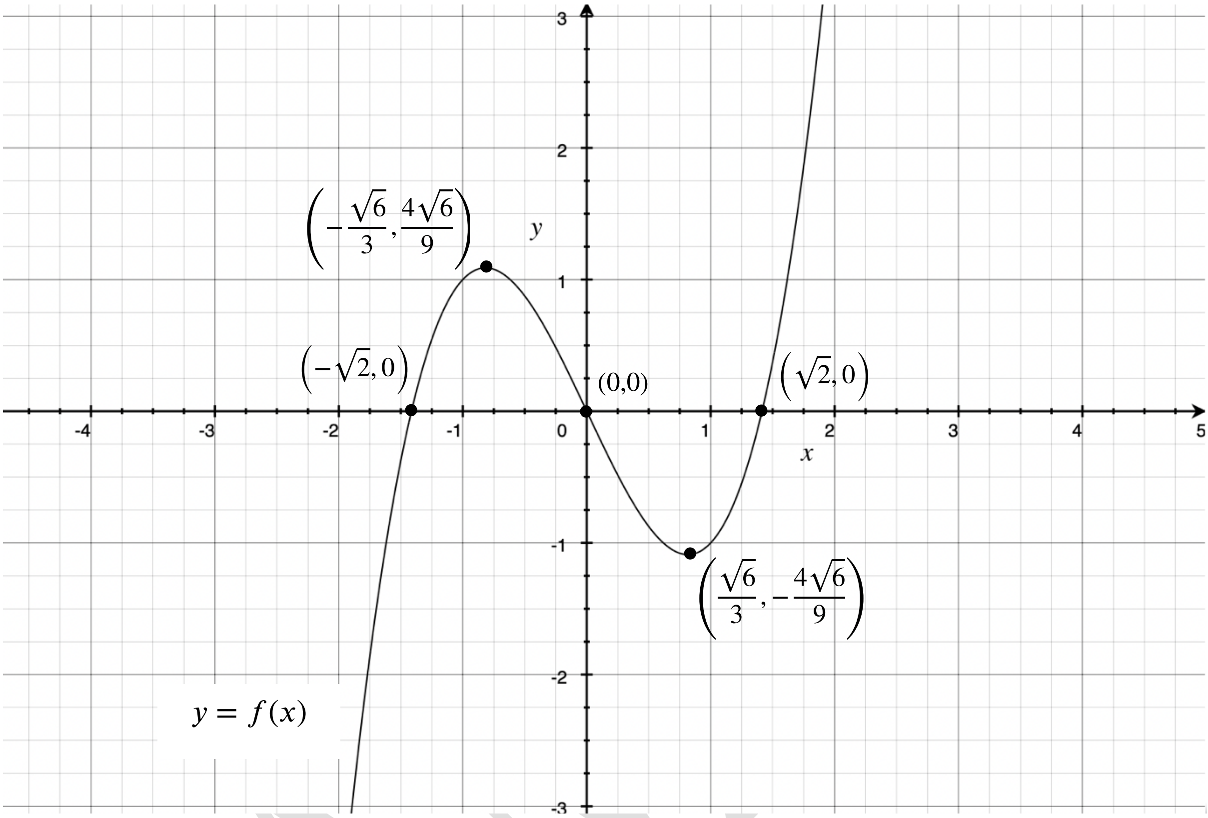
We can also note that f is an odd function, satisfying $f(-a) = -f(a)$, so the turning points of f satisfy $(b, f(b)), (-b, -f(b))$.

Therefore, we can find both turning points by finding the coordinates of just one turning point in this case.

e. Sketch the graph of $y = f(x)$ on the axes below. Label any stationary points and axes intercepts with their coordinates.

2 marks

Solution:



Let $g: R \to R, g(x) = x^3 - ax$ where $a \in R$.

f. Find the value(s) of a such that the line $y = x$ is tangent to the graph of g.

3 marks

Solution:

- Find the values of x in terms of a such that $\frac{d}{dx}(\boldsymbol{x}) = \boldsymbol{g'(x)}$

  $g'(x) = 3x^2 - a$ and $\frac{d}{dx}(x) = 1$

  $3x^2 - a = 1$

  $x = \pm\sqrt{\frac{1+a}{3}}$

- Solve $\boldsymbol{g(x) = x}$ for the values of x above.

  $$g\left(\sqrt{\frac{1+a}{3}}\right) = \left(\sqrt{\frac{1+a}{3}}\right)^3 - a\left(\sqrt{\frac{1+a}{3}}\right) = \sqrt{\frac{1+a}{3}}$$

  $$\sqrt{\frac{1+a}{3}}\left(\frac{1+a}{3}\right) - a\left(\sqrt{\frac{1+a}{3}}\right) = \sqrt{\frac{1+a}{3}}$$

  $a = -1$ is the only solution to the above equation.

  $$g\left(-\sqrt{\frac{1+a}{3}}\right) = \left(-\sqrt{\frac{1+a}{3}}\right)^3 - a\left(-\sqrt{\frac{1+a}{3}}\right) = -\sqrt{\frac{1+a}{3}}$$

  $$\sqrt{\frac{1+a}{3}}\left(\frac{1+a}{3}\right) - a\left(\sqrt{\frac{1+a}{3}}\right) = \sqrt{\frac{1+a}{3}}$$

  This is the same equation as the one above, giving $a = -1$

- Concluding **statement.**

  Therefore, $a = -1$ is the only value of a such that g is tangent to $y = x$.

The equation $\sqrt{\frac{1+a}{3}}\left(\frac{1+a}{3}\right) - a\left(\sqrt{\frac{1+a}{3}}\right) = \sqrt{\frac{1+a}{3}}$ can be reduced to $\frac{1+a}{3} - a = 1$ by dividing by $\sqrt{\frac{1+a}{3}}$, provided that $\sqrt{\frac{1+a}{3}} \neq 0$.

As $\frac{1+a}{3} - a = 1$ leads to $a = -1$ and $\sqrt{\frac{1+a}{3}} = 0$ leads to $a = -1$, we can say that these equations are consistent, and thus $a = -1$ only.